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ASYMPTOTICS OF CORRELATION FUNCTION OF TWIST FIELDS IN TWO DIMENSIONAL LATTICE FERMION MODEL

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Abstract

In two-dimensional lattice fermion model a determinant representation for the two-point correlation function of the twist field in the disorder phase is obtained. This field is defined by twisted boundary conditions for lattice fermion field. The large distance asymptotics of the correlation function is calculated at the critical point and in the scaling region. The result is compared with the vacuum expectation values of exponential fields in the sine-Gordon model conjectured by S.Lukyanov and A.Zamolodchikov.

1. Introduction

During the last time the progress has been made in calculation of the long distance asymptotics of a correlation functions of local fields in some integrable two-dimensional quantum field theories [1, 2]. Usually in these theories two-point correlation functions of local fields can be represented as an infinite series of the form-factor contributions, which are calculated using a method of the form-factor bootstrap [3] or the angular quantization [4, 5]. In [6, 7] a summation method of the form-factor decomposition of the correlation function was developed. This method allows to obtain a closed expression for the correlation function through Fredholm determinant of the integral operator and to do analysis of asymptotic behaviour of one.

In this paper we analyse a large distance behaviour of the two-point correlation function of the twist field $\mu_\nu(r)$ in the disorder phase of two-dimensional lattice fermion model. At $\nu = \frac{1}{2}$ this field coincides with the disorder field $\mu(r)$ and the correlation function of these fields satisfies the following relation [8, 9]:

$$\langle \mu(0)\mu(r)\rangle = \langle \mu^I(0)\mu^I(r)\rangle^2, \tag{1.1}$$

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where $\mu^I(r)$ is a disorder field in the Ising model.

Using the functional integral method in two-dimensional lattice fermion model, we obtain a determinant representation of the correlation function. Calculating the asymptotics of the correlation function for $r \to \infty$, we find the vacuum expectation values $\langle \mu_\nu(r) \rangle$. Here index ν ($0 < \nu < 1$) denotes twisted boundary conditions for lattice fermion field along the correlation line. In scaling limit (massive fermion quantum field theory) this field is massive analog of the twist field in two dimensional conformal field theory with central charge $c = 1$ [10].

In the free fermion point of the sine-Gordon model the vacuum expectation value $< \mu(r) >$ is connected by simple relation with the vacuum expectation value of the exponential field $\exp(i\nu\phi)$ for $\nu = \frac{1}{2}$. The sine-Gordon model is described by the Euclidean action

$$S = \frac{1}{4\pi} \int dx d\tau \left(\partial_k \phi \partial_k \phi + \mu \cos(\hat{\beta}\phi) \right).$$

The free fermion point occurs for $\hat{\beta} = 1$.

In this point the exponential field can be defined by the braiding relations with the free fermion field [11]:

$$\begin{aligned} \mathrm{e}^{i\nu\phi(\tau,x)} \psi(\tau, y) &= \psi(\tau, y) \mathrm{e}^{i\nu\phi(\tau,x)}, \quad \text{if } y < x, \\ \mathrm{e}^{i\nu\phi(\tau,x)} \psi(\tau, y) &= \mathrm{e}^{2\pi i\nu} \psi(\tau, y) \mathrm{e}^{i\nu\phi(\tau,x)}, \quad \text{if } y > x. \end{aligned} \tag{1.2}$$

As it was shown in [8, 9, 12] the correlation functions of the disorder fields satisfy the following relations

$$\langle \mu(0)\mu(r) \rangle = \langle \cos \tfrac{1}{2}\phi(0) \, \cos \tfrac{1}{2}\phi(r) \rangle. \tag{1.3}$$

From here one gets

$$\langle \mu(0)\mu(r) \rangle >= \frac{1}{2} \langle \mathrm{e}^{i\frac{1}{2}\phi(0)} \, \mathrm{e}^{i\frac{1}{2}\phi(r)} \rangle + \frac{1}{2} \langle \mathrm{e}^{i\frac{1}{2}\phi(0)} \, \mathrm{e}^{-i\frac{1}{2}\phi(r)} \rangle. \tag{1.4}$$

Using this equation, we can obtain a relation between the vacuum expectation values of the disorder and exponential fields.

Note that for calculation of the left and right hand sides of (1.4) it is nessesary to regularize explicit expressions for the correlation functions that it leads to appearence of the corresponding wave renormalization constants. To be rid of them at calculation of the expectation values it is convenient to normalize the correlation functions on behaviour for $r \to 0$. In this limit we obtain from (1.4)

$$\langle \mu(0)\mu(r) \rangle = \frac{1}{2} \langle \mathrm{e}^{i\frac{1}{2}\phi(0)} \, \mathrm{e}^{-i\frac{1}{2}\phi(r)} \rangle, \tag{1.5}$$

where we have used the following asymptotics for the correlation function of the exponential fields for $r \to 0$ [13]

$$\langle \mathrm{e}^{i\nu\phi(0)} \, \mathrm{e}^{i\nu'\phi(r)} \rangle = \langle \mathrm{e}^{i(\nu+\nu')\phi} \rangle r^{2\nu\nu'}. \tag{1.6}$$

Then for a ratio of (1.4) in the limit $r \to \infty$ and (1.5) one gets

$$\frac{\langle \mu(0)\mu(r) \rangle > |_{r\to\infty}}{\langle \mu(0)\mu(r) \rangle|_{r\to 0}} = \frac{\langle \mu \rangle^2}{\langle \mu(0)\mu(r) \rangle|_{r\to 0}} = \frac{2 \langle \mathrm{e}^{i\frac{1}{2}\phi} \rangle^2}{\langle \mathrm{e}^{i\frac{1}{2}\phi(0)} \, \mathrm{e}^{-i\frac{1}{2}\phi(r)} \rangle|_{r\to 0}}. \tag{1.7}$$

As it will shown below one can choose such normalization of the disorder fields that asymptotic behaviour of the correlation functions in denominators of the left and right hand sides of (1.7) coincides. In this case we obtain from (1.7) $\langle\mu\rangle = \sqrt{2}\langle\exp(i\frac{1}{2}\phi)\rangle$. In the paper [13] (also see [14]) it was conjectured the following expression for the vacuum expectation values of the exponential fields ($|\nu| < 1$)

$$\langle \mathrm{e}^{i\nu\phi}\rangle = \left(\frac{m}{2}\right)^{\nu^2} \exp\left[\int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2\mathrm{e}^{-t}\right)\right]. \tag{1.8}$$

This expression it is easy to obtain by means of results of the paper [21]. Here the asymptotic behaviour of the correlation function of the exponential fields for $r \to 0$ was calculated. Comparing our result (4.20) and (1.8), we obtain

$$\langle\mu_\nu\rangle = 4^{\nu^2}\langle\exp(i\nu\phi)\rangle. \tag{1.9}$$

Emphasize that this relation is correct for such normalization of the fields that the correlation functions of the twist and exponential fields have the same asymptotics for $r \to 0$

$$\langle\mu_\nu(0)\,\mu_\nu(r)\rangle \underset{r\to 0}{=} < \mathrm{e}^{i\nu\phi(0)}\,\mathrm{e}^{-i\nu\phi(r)} > \underset{r\to 0}{=} \frac{1}{r^{\,2\nu^2}}.$$

In section 2 the determinant representation of the two-point correlation function of the twist field is derived. In section 3 asymptotic behaviour of the correlation function are considered and new determinant representation through the matrix of the Toeplitz type are obtained in the limit $r \to \infty$. In section 4, using this representation, asymptotics of the correlation function is calculated. In Appendices some relations used in this paper are derived.

2. Functional integral representation of the correlation function

It is known [15, 16], that the correlation function $\langle\mu^I(0)\mu^I(r)\rangle$ in the two dimensional Ising model can be represented in the form of the functional integral for the lattice Majorana fermion theory with antiperiodic boundary conditions for the fermion field with the exception of the line $[0, r-1]$, where the fermion field has periodic boundary conditions.

The same way let us define the two-point correlation function of the field $\mu_\nu(r)$ in the lattice Dirac fermion theory through a ratio of the following functional integrals

$$\langle\mu_\nu(0)\mu_\nu(r)\rangle = \frac{\int d[\psi\bar{\psi}]\mathrm{e}^{S_r[\psi]}}{\int d[\psi\bar{\psi}]\mathrm{e}^{S[\psi]}}, \tag{2.1}$$

with the action

$$S_r[\psi] = (\bar{\psi}\hat{D}\psi) = \sum_{\rho,\rho'}\bar{\psi}(\rho)D_{\rho,\rho'}\psi(\rho'), \tag{2.2}$$

where $\psi(\rho)$ is a complex two component grassmann field; the coodinates of the lattice sites $\rho = (x, y)$ run through the values $x = 1, ..., n_x$, $y = 1, ..., n_y$; $\hat{D}$ is the lattice Dirac operator

$$\hat{D} = \begin{pmatrix} u & v \\ -v^T & u^T \end{pmatrix} = \begin{pmatrix} 1 - t\nabla_x & 1 - t\nabla_y \\ -(1 - t\nabla_{-y}) & 1 - t\nabla_{-x} \end{pmatrix}. \tag{2.3}$$

Here ∇_x, ∇_y denote the shift operators along the X and Y axes:

$$\nabla_x \psi(\rho) = \psi(\rho + \hat{x}), \quad \nabla_y \psi(\rho) = \psi(\rho + \hat{y}),$$

where $\hat{x}$, $\hat{y}$ are unit vectors. Index "r" denotes that in action $S_r[\psi]$ operator ∇_y has twisted boundary conditions along the line $[0, r-1]$ $(r - 1 = (r - 1, 0))$:

$$\nabla_y \psi(x, n_y) = -\mathrm{e}^{2\pi i\nu} \psi(x, 0), \quad x = 0, 1, \ldots, r - 1.$$

In denominator (2.1) the action $S[\psi]$ coincides with $S_r[\psi]$ by the form, but ∇_y satisfies antiperiodic boundary conditions along all boundary.

It is not hard to check that the operator $\hat{D}$ satisfies the relation $\hat{D}^{-1} = \hat{D}^T \cdot \hat{K}$, where $\hat{K}$ is the lattice Klein-Gordon operator

$$\hat{K} = \det \hat{D} = uu^T + vv^T = 2(1 - t)^2 - t[(\nabla_x - 2 + \nabla_{-x}) + (\nabla_y - 2 + \nabla_{-y})].$$

In the "naive" continuum limit (the lattice constant $a \to 0$) $\hat{K} = a^2 t(m^2 - \partial_i \partial_i)$, where $\partial_i = (\nabla_i - 1)/a$ and $ma/\sqrt{2} = t^{-\frac{1}{2}} - t^{\frac{1}{2}}$. From here the critical point of the model is $t_c = 1$, and the scaling region is determined by relation $|t_c - t| = ma/\sqrt{2}$. In momentum representation the operator $\hat{K}$ is diagonal and the function

$$K(p) = 2(1 - t)^2 + 4t\Big(\sin^2 \frac{p_x}{2} + \sin^2 \frac{p_y}{2}\Big)$$

has unique minimum at $p_x = p_y = 0$ in the Brillouin zone and unlike the usual lattice Dirac operator we have not problem with the fermion dubling.

In order to integrate over the fermion fields in the numerator (2.1), in the action $S_r[\psi]$ it is convenient to go to operator ∇_y with antiperiodic boundary conditions along all boundary. After that we get additional term $\delta S_r[\psi]$ in the action $S_r[\psi]$, which contains a lattice defect:

$$S_r[\psi] = S[\psi] + \delta S_r[\psi],$$

where

$$\delta S_r[\psi] = t \sum_{x=0}^{r-1} [-\xi \bar{\psi}^1(x, n_y) \psi^2(x, 0) + \xi^* \bar{\psi}^2(x, 0) \psi^1(x, n_y)] = (\bar{\psi} \hat{P}^T \hat{\Xi} \hat{P} \psi).$$

When

$$\langle \mu_\nu(0) \mu_\nu(r) \rangle = \langle \mathrm{e}^{\delta S_r[\psi]} \rangle = \frac{\int d[\psi\bar{\psi}] \mathrm{e}^{S[\psi] + \delta S_r[\psi]}}{\int d[\psi\bar{\psi}] \mathrm{e}^{S[\psi]}}. \tag{2.4}$$

Here the projective operator $\hat{P}$ selects lattice sites on the line $[0, r-1]$

$$\hat{P}_{l,x} = \delta_{l,x} \delta_{n_y, y} \begin{pmatrix} 1 & 0 \\ 0 & \nabla_y \end{pmatrix}, \quad l = 0, 1, \ldots, r - 1;$$

and

$$\hat{\Xi} = t \begin{pmatrix} 0 & -\xi \hat{I}^{(r)} \\ \xi^* \hat{I}^{(r)} & 0 \end{pmatrix},$$

where $\xi = 1 - \mathrm{e}^{2\pi i\nu}$ and $\hat{I}^{(r)}$ is a unit matrix of the size $r \times r$.

For calculation of the functional integrals in (2.4) it is convenient to represent $\mathrm{e}^{\delta S_r[\psi]}$ through the integral over a auxiliary field

$$\mathrm{e}^{\delta S_r[\psi]} = |\hat{\Xi}| \int d[\chi\bar{\chi}] \exp[(\bar{\chi}\hat{\Xi}^{-1}\chi) + (\bar{\chi}\hat{P}\psi) + (\bar{\psi}\hat{P}^T\chi)]. \tag{2.5}$$

Using(2.5) and integrating over $\psi(\rho)$ in (2.4), one gets

$$\begin{aligned} \langle \mu_\nu(0)\mu_\nu(r)\rangle &= |\hat{\Xi}| \int d[\chi\bar{\chi}] \exp[(\bar{\chi}\hat{\Xi}^{-1}\chi) + (\bar{\chi}\hat{P}\hat{D}^{-1}\hat{P}^T\chi)] = \\ &= |\hat{\Xi}| \cdot |\hat{\Xi}^{-1} + \hat{P}\hat{D}^{-1}\hat{P}^T| = |\hat{G}|, \end{aligned} \tag{2.6}$$

where $|\hat{G}|$ denotes the determinat of a block matrix of the dimension (2×2) with blocks of the dimension $(r \times r)$. The matrix $\hat{G}$ has the form

$$\hat{G} = \begin{pmatrix} g^{11}_{x,x'} \sin \pi\nu & -g^{12}_{x,x'} \sin \pi\nu + iI^{(r)}_{x,x'} \cos \pi\nu \\ g^{12}_{x,x'} \sin \pi\nu + iI^{(r)}_{x,x'} \cos \pi\nu & (g^{11}_{x,x'})^T \sin \pi\nu \end{pmatrix}, \tag{2.7}$$

where

$$g^{11}_{x,x'} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \frac{d^2p}{K(p)} \mathrm{e}^{ip_x(x-x')} 2tu^*(p_x), \quad u(p_x) = 1 - t\mathrm{e}^{ip_x},$$

$$g^{12}_{x,x'} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \frac{d^2p}{K(p)} \mathrm{e}^{ip_x(x-x')} 2t(u^*(p_x) + u(p_x)).$$

Let us show that the correlation function (2.6) for $\nu = \frac{1}{2}$ is equal to square of the correlation function of the disorder field in the Ising model. For this it is necessary to do a similarity transformation of the matrix $\hat{G}$ in (2.6) by means of the unitarity matrix $\hat{Q}$

$$\hat{Q} = \mathrm{e}^{i\gamma_5 \frac{\theta}{2}}, \quad \gamma_5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathrm{ctg}\theta = t, \tag{2.8}$$

As result we obtain

$$G(r) = \langle \mu_\nu(0)\mu_\nu(r)\rangle \equiv |\hat{Q} \cdot \hat{G} \cdot \hat{Q}^{-1}| = |\cos^2 \pi\nu + \sin^2 \pi\nu \hat{V} \cdot \hat{V}^T|, \tag{2.9}$$

where

$$V_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-(x-x')} V(z), \quad V(z) = \sqrt{\frac{(1-\alpha z)(1-\beta z^{-1})}{(1-\alpha z^{-1})(1-\beta z)}}, \tag{2.10}$$

$$\alpha = t\frac{\sqrt{1+t^2}+t}{\sqrt{1+t^2}+1}, \quad \beta = t\frac{\sqrt{1+t^2}-t}{\sqrt{1+t^2}+1}. \tag{2.11}$$

Here the matrix $V_{x,x'}$ exactly coincides with classical expression for the Toeplitz matrix [17], the determinant of which determines the correlation function $\langle \mu^I(0)\mu^I(r)\rangle$ in the paramagnetic phase of the Ising model. It is obvious that (1.1) follows from (2.9) for $\nu = \frac{1}{2}$. In further we include coefficient $(2\pi i)^{-1}$ in the integration measure of contour integrals.

Note that the transformation $\hat{Q}$ is connected with particular version of the lattice Dirac operator (2.3). In the "naive" continuum limit we obtain for the action (2.2)

$$S[\psi] = \int d^2\rho \bar{\psi}(\rho)(me^{i\gamma_5\frac{\pi}{4}} - \gamma_i\partial_i)\psi(\rho), \tag{2.12}$$

where

$$\gamma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

At the critical point $t_c = 1$ the angle θ in (2.8) is equal to $\pi/4$ and the transformation $\hat{Q}$ is the γ_5-rotation of the Grassmann field: $\psi' = \exp(i\gamma_5\pi/8)\psi$, after that the Dirac action takes the usual form

$$S[\psi] = \int d^2\rho \bar{\psi}(\rho)(m - \gamma_i\partial_i)\psi(\rho). \tag{2.13}$$

3. Toeplitz determinant representation of the correlation function

In this section we show that in the scaling limit ($ma \to 0$) at $r \to \infty$ a evaluation of the correlation function (2.9) can be reduced to a calculation of the Toeplitz determinant:

$$G(r) = |\cos^2\pi\nu + \sin^2\pi\nu\hat{V}\cdot\hat{V}^T| \underset{r\to\infty}{=} |V^{(\nu)}\cdot V^{(\nu)^T}| = |V^{(\nu)}|^2, \tag{3.1}$$

where

$$V^{(\nu)}_{x,x'} = \oint\limits_{|z|=1} \frac{dz}{z} z^{-(x-x')}V^{(\nu)}(z), \quad V^{(\nu)}(z) = \left[\frac{(1-\alpha z)(1-\beta z^{-1})}{(1-\alpha z^{-1})(1-\beta z)}\right]^{\nu} \tag{3.2}$$

is a Toeplitz matrix. For $\nu = \frac{1}{2}$ the kernel $V^{(\nu)}(z)$ coincides with the kernel $V(z)$ (2.10).

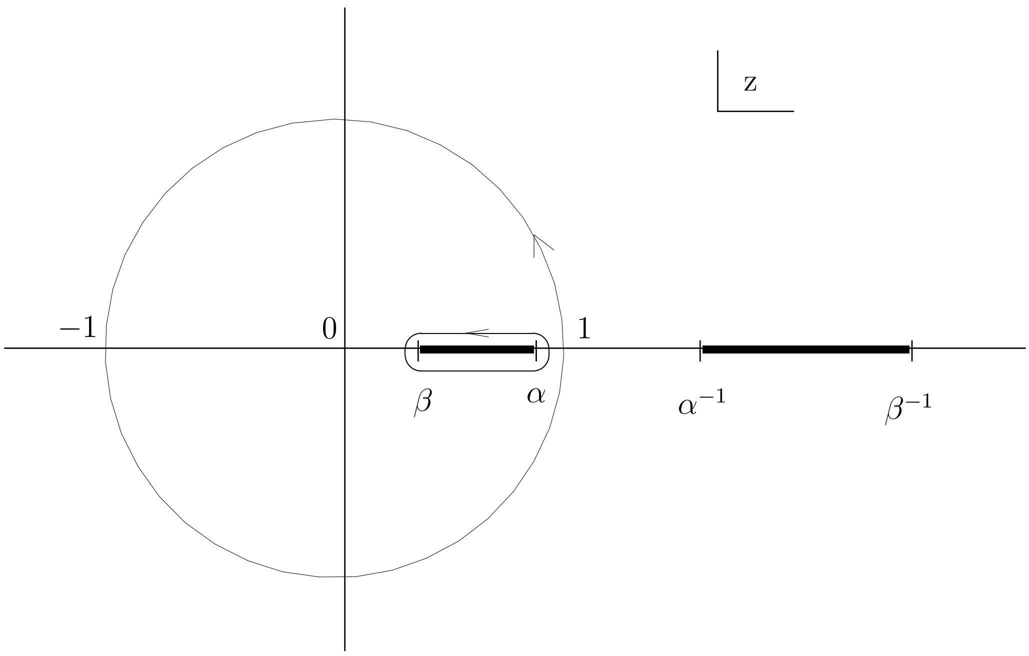
In Appendix A it is shown that the correlation function (2.9) can be represented in the following form

$$G(r) \underset{r\to\infty}{=} |1 - \sin^2\pi\nu\hat{A}|^2, \tag{3.3}$$

where

$$A_{x,x'} = \oint\limits_{|z_1z_2|<1} \frac{dz_1dz_2(z_1)^x(z_2)^{x'}}{(1-z_1z_2)} V^{-1}(z_1)V^{-1}(z_2).$$

Here the integration contour passes between the cuts depicted on Fig. 1.



$Fig.1.$

Using (3.3), we get

$$\tfrac{1}{2}\ln G(r) = \mathrm{Sp}\ln(1-\sin^2\pi\nu\hat{A}) = -\sum_{k=1}^{\infty}\frac{a_k}{k}\left(\frac{\sin\pi\nu}{\pi}\right)^{2k}, \tag{3.4}$$

where

$$a_k = \pi^{2k}\mathrm{Sp}\hat{A}^k = \pi^{2k}\oint\prod_{l=1}^{2k}dz_l\frac{\prod_{l=1}^{k}[1-(z_lz_{l+1})^r]}{\prod_{l=1}^{2k}[(1-z_lz_{l+1})V(z_l)]}, \quad z_{2l+1}=z_1. \tag{3.5}$$

For $r\to\infty$ the terms in the right-hand side of (3.5), which contain the product $(z_lz_{l+1})^r$, are exponentially small, so that

$$a_k = \pi^{2k}\oint\frac{\prod_{l=1}^{2k}dz_l}{\prod_{l=1}^{2k}[(1-z_lz_{l+1})V(z_l)]} = \int_{\beta}^{\alpha}\frac{\prod_{l=1}^{2k}dz_l}{\prod_{l=1}^{2k}(1-z_lz_{l+1})}\prod_{l=1}^{2k}\left[\frac{(\alpha-z_l)(1-\beta z_l)}{(1-\alpha z_l)(z_l-\beta)}\right]^{\frac{1}{2}}. \tag{3.6}$$

At $ma=0$ ($\alpha=1$) the integral in (3.6) has logarithmic divergence on the upper limit (in a neighborhood of the "upper right cone" of the $2k$-dimensional hypercube with the coordinates $z_l=1$). Let us isolate and evaluate this divergence:

$$a_k \simeq \int_{\alpha-\varepsilon}^{\alpha}\prod_{l=1}^{2k}\frac{dz_l}{1-z_lz_{l+1}}\prod_{l=1}^{2k}\left(\frac{\alpha-z_l}{1-\alpha z_l}\right)^{\frac{1}{2}} \simeq \int_{1}^{\varepsilon/ma}\prod_{l=1}^{2k}\frac{dz_l}{z_l+z_{l+1}}. \tag{3.7}$$

For derivation of this estimate we did the following substitutions of the integration variables: $z_l=\alpha-z'_l$, $z'_l=(ma)z''_l$, $z''_l=z_l+1$. Isolating the logarithmic singularity in the last integral, we obtain

$$a_k = \int\limits_1^{\varepsilon/ma} \prod_{l=1}^{2k} \frac{dz_l}{z_l + z_{l+1}} = -kb_k \ln(ma) + \text{const.} \tag{3.8}$$

Differentiating over ma the left- and right-hand sides of (3.8), the coefficients b_k can be expressed through the multiple integrals

$$b_k = 2 \int\limits_0^1 \prod_{l=1}^{2k-1} dz_l \cdot \left[(1+z_1)(1+z_{2k-1}) \prod_{l=1}^{2k-2} (z_l + z_{l+1})\right]^{-1}. \tag{3.9}$$

Using (3.8) and (3.4), we get the asymptotic evaluation of the correlation function for $r \to \infty$, $ma \to 0$

$$\tfrac{1}{2} \ln G(\infty) = \ln(ma) \sum_{k=1}^{\infty} b_k \left(\frac{\sin \pi\nu}{\pi}\right)^{2k} + \text{const.} \tag{3.10}$$

The integrals in the first three coefficients in (3.9) can be reduced to the table ones

$$b_1 = 1, \quad b_2 = 2\zeta(2), \quad b_3 = 4\zeta^2(2) + 6\zeta(4), \tag{3.11}$$

where $\zeta(k)$ is the Rieman ζ-function

$$\zeta(k+1) = \frac{(-1)^k}{k!} \int\limits_0^1 \frac{dx \ln^k x}{1-x}.$$

Decomposing in (3.10) the functions $(\sin \pi\nu/\pi)^{2k}$ over powers of ν and using the values (3.11) for the coefficients b_k, it is not difficult to get

$$\begin{gathered} b_1 \frac{\sin^2 \pi\nu}{\pi^2} = \nu^2 + \mathrm{O}(\nu^4), \\ b_1 \frac{\sin^2 \pi\nu}{\pi^2} + b_2 \frac{\sin^4 \pi\nu}{\pi^4} = \nu^2 + \mathrm{O}(\nu^6), \\ b_1 \frac{\sin^2 \pi\nu}{\pi^2} + b_2 \frac{\sin^4 \pi\nu}{\pi^4} + b_3 \frac{\sin^6 \pi\nu}{\pi^6} = \nu^2 + \mathrm{O}(\nu^8), \end{gathered} \tag{3.12}$$

i.e. every following term in the series (3.11) annihilates more high powers of ν^{2k}.

Although the integrals (3.9) is hardly calculated for multiplicity higher five, one can assume that in more high orders relations of the type (3.12) also take place. Then

$$\sum_{k=1}^{\infty} b_k \left(\frac{\sin \pi\nu}{\pi}\right)^{2k} = \nu^2. \tag{3.13}$$

It means that the generation function for the coefficients b_k has the form

$$\frac{1}{\pi^2} \arcsin^2(\pi\sqrt{z}) = \sum_{k=1}^{\infty} b_k z^k. \tag{3.14}$$

Then for the determinant (3.1) we obtain

$$G(\infty) \underset{(ma)\to 0}{=} \mathrm{const}\cdot (ma)^{2\nu^2}. \tag{3.15}$$

Note that (3.14) allows to find values of the integrals (3.9) of arbitrary multiplicity.

Let us prove the asymptotics (3.15). The following observation is key for the proof (it follows from the calculations in (3.7)): the asymptotic evaluation ($ma \to 0$) of the integrals in (3.6) is insensitive to the exponent in the expression for kernel of the matrix $V_{x,x'}$. It allows to introduce the new matrix $V^{(\nu)}_{x,x'}$ with the kernel (3.2). Let us define the correlation function

$$G^{(\nu)}(r) = |\hat{V}^{(\nu)}\cdot \hat{V}^{(\nu)\,T}|, \tag{3.16}$$

where the matrix $\hat{V}^{(\nu)}$ is determinated in (3.2). Using the results of Appendix A, (3.16) can be represented as

$$G^{(\nu)}(r) \underset{r\to\infty}{=} |1-\hat{A}^{(\nu)}|^2, \tag{3.17}$$

where the matrix $\hat{A}^{(\nu)}$ has the form

$$A^{(\nu)}_{x,x'} = \oint \frac{dz_1 dz_2 (z_1)^x (z_2)^{x'}}{(1-z_1z_2)}\frac{1}{V^{(\nu)}(z_1)V^{(\nu)}(z_2)}. \tag{3.18}$$

The trace of the k-th power of the matrix $\hat{A}^{(\nu)}$ is expressed through the $2k$-multiple integral

$$\mathrm{Sp}[(\hat{A}^{(\nu)})^k] = \left(\frac{\sin\pi\nu}{\pi}\right)^{2k}\int\limits_{\beta}^{\alpha}\prod_{l=1}^{2k}\frac{dz_l}{(1-z_lz_{l+1})}\prod_{l=1}^{2k}\left[\frac{(\alpha-z_l)(1-\beta z_l)}{(1-\alpha z_l)(z_l-\beta)}\right]^{\nu}\prod_{l=1}^{k}[1-(z_lz_{l+1})^r]. \tag{3.19}$$

Here, unlike (3.4), the factor $(\sin\pi\nu)^{2k}$ appeares on account of the exponent ν at contraction of the integration contour. Taking into account (3.19) for $r\to\infty$, we get the expression for $G^{(\nu)}(r)$ which is similar to (3.4)

$$\tfrac{1}{2}\ln G^{(\nu)}(r) = \mathrm{Sp}\ln(1-\hat{A}^{(\nu)}) = -\sum_{k=1}^{\infty}\frac{a_k^{(\nu)}}{k}\left(\frac{\sin\pi\nu}{\pi}\right)^{2k}, \tag{3.20}$$

where

$$a_k^{(\nu)} = \int\limits_{\beta}^{\alpha}\prod_{l=1}^{2k}\frac{dz_l}{(1-z_lz_{l+1})}\prod_{l=1}^{2k}\left[\frac{(\alpha-z_l)(1-\beta z_l)}{(1-\alpha z_l)(z_l-\beta)}\right]^{\nu}\prod_{l=1}^{k}[1-(z_lz_{l+1})^r]. \tag{3.21}$$

For $r\to\infty$ the terms in (3.21) containing the power of $(z_lz_{l+1})^r$ disappear. Although the rest integrals in (3.21) diverge on the upper limit the difference between (3.6) and (3.21)

$$\begin{aligned}\Delta a_k^{(\nu)} &\equiv a_k - a_k^{(\nu)} = \\ &= \int\limits_{\beta}^{\alpha}\prod_{l=1}^{2k}\frac{dz_l}{(1-z_lz_{l+1})}\left\{\prod_{l=1}^{2k}\left[\frac{(\alpha-z_l)(1-\beta z_l)}{(1-\alpha z_l)(z_l-\beta)}\right]^{\frac{1}{2}} - \prod_{l=1}^{2k}\left[\frac{(\alpha-z_l)(1-\beta z_l)}{(1-\alpha z_l)(z_l-\beta)}\right]^{\nu}\right\}\end{aligned} \tag{3.22}$$

is finite for $\alpha = 1$

$$\Delta a_k^{(\nu)} = \sqrt{2}(\tfrac{1}{2}-\nu)\int\limits_{1-\varepsilon}^{1}\prod_{l=1}^{2k}\frac{dz_l}{(1-z_lz_{l+1})}\sum_{l=1}^{2k}(1-z_l) + \mathrm{const} = \mathrm{const}.$$

Hence a ratio of the determinants (2.9) and (3.16) is also finite and it is not equal zero

$$\gamma \equiv \frac{G(\infty)}{G^{(\nu)}(\infty)} = \frac{|\cos^2 \pi\nu + \sin^2 \pi\nu \hat{V} \cdot \hat{V}^T|}{|\hat{V}^{(\nu)} \cdot \hat{V}^{(\nu)\,T}|} \underset{(ma)\to 0}{=} \text{const.} \tag{3.23}$$

Therefore the singular over (ma) factor in $G(\infty)$ coincides with one in the determinant (3.16). Since the matrix $\hat{V}^{(\nu)}$ is a Toeplitz matrix we can calculate (2.9) in limit $r \to \infty$, $(ma) \to 0$.

Note that we have considered the ratio (3.23) in limit $r = \infty$, $(ma) \to 0$. In the paper [13] the correlation function of the exponential field was normalized on the correlation function in the massless fermion field theory. Therefore for comparison of our results with [13] it is necessary to evaluate the ratio (3.23) in limit $(ma) = 0$, $r \to \infty$: in other words we must verify that the value of γ does not depend on a sequence of the limits.

It can show that at $(ma) = 0$ the difference of the coefficients (3.5) and (3.21) differs from (3.22) on some term decreasing with increase of r

$$a_k - a_k^{(\nu)} = \Delta a_k + \frac{\phi_k(r)}{r^2}.$$

For $\phi_k(r)$ we have the following asymptotic evaluation

$$\phi_k(r) \simeq \frac{\ln^k r}{k!}. \tag{3.24}$$

As a result we get at $ma = 0$ and $r \to \infty$

$$\frac{G(r)}{G^{(\nu)}(r)} = \gamma \exp\Big[\frac{1}{r^2} \sum_{k=1}^{\infty} \phi_k(r)\Big]. \tag{3.25}$$

Using (3.24), it is not hard to sum the series in the exponent of the right-hand side of (3.25) and to obtain the following estimate

$$\frac{G(r)}{G^{(\nu)}(r)} = \gamma + \mathrm{O}(1/r). \tag{3.26}$$

4. Asymptotics of the correlation function

In this section we calculate the large distance behaviour of the correlation function $G(r)$. We use the determinant representation (3.1) for it in this case. Since in (3.1) the matrix $V_{x,x'}^{(\nu)}$ is a matrix of the Toeplitz type it can apply the McCoy and Wu techniques [17], which they used for the asymptotic evaluation of the correlation function in the Ising model ($\nu = \frac{1}{2}$). This techniques requires some generalization for $0 < \nu < 1$. In Appendix B we obtained the following asymptotic expression for the determinant of the matrix $\widehat{V}^{(\nu)}$ (the formula B.24):

$$\ln|\widehat{V}^{(\nu)}| \underset{r\to\infty}{=} (r+1) \oint \frac{dz}{z} \ln V^{(\nu)} - \oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}). \tag{4.1}$$

Substituting here the explicit expressions for $V^(\nu)(z)$, $P(z)$ and $Q(z)$:

$$V^{(\nu)}(z) = P(z)Q(z^{-1}), \quad P(z) = \left(\frac{1-\alpha z}{1-\beta z}\right)^{\nu}, \; Q(z) = \left(\frac{1-\beta z}{1-\alpha z}\right)^{\nu} = \frac{1}{P(z)}, \tag{4.2}$$

one gets

$$\oint \frac{dz}{z} \ln V^{(\nu)} = \nu \left(\oint \frac{dz}{z} \ln P(z) - \oint \frac{dz}{z} \ln P(z^{-1}) \right) = \nu \left(P(0) - P(0) \right) = 0,$$

$$-\oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}) = \nu^2 \oint dz \ln P(z) \left(\frac{1}{z-\alpha} - \frac{1}{z-\beta} \right) = \mu^2 \left(\ln P(\alpha) - \ln P(\beta) \right).$$

Thus

$$|\widehat{V}^{(\nu)}| \underset{r\to\infty}{=} \left(\frac{P(\alpha)}{P(\beta)}\right)^{\nu^2} = \left[\frac{(1-\alpha^2)(1-\beta^2)}{(1-\alpha\beta)}\right]^{\nu^2}. \tag{4.3}$$

Note that this expresion is derived for finite values of (ma) and r. The derivation of (4.1) is also correct in the scaling region: $(ma) \to 0$, $r \to \infty$, $r(ma) = \text{const}$. However for $(ma) = 0$, $r = \text{const}$ the integrals in the right-hand side of (4.1) are divergent and the calculation of $|\hat{V}^{(\nu)}|$ at $(ma) = 0$ and $r \to \infty$ requires the special consideration.

Note that the following ratio of determinants

$$\frac{|\hat{V}^{(\nu)}(\beta)|}{|\hat{V}^{(\nu)}(0)|} \underset{r\to\infty}{=} \left[\frac{1-\beta^2}{(1-\alpha\beta)^2}\right]^{\nu^2} \underset{(ma)=0}{=} 2^{\nu^2/2}. \tag{4.4}$$

is finite both in the scaling regime and at the critical point. Here we explicitly indicated a dependent of the matrix $\hat{V}^{(\nu)}$ on the parameter β (the matrix $\hat{V}^{(\nu)}(0)$ is determined by the kernel (3.2) at $\beta = 0$).

The relation (4.4) it is not hard to obtain by means of (4.1). Really, for the ratio of the determinants in the left-hand side of (4.4) we get

$$\begin{aligned}
\ln\left(\frac{|\hat{V}^{(\nu)}(\beta)|}{|\hat{V}^{(\nu)}(0)|}\right) &= -\oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}) + \\
&\quad + \oint dz \left[\ln P(z) + \nu \ln(1-\beta z)\right] \frac{\partial}{\partial z} \left[\ln Q(z^{-1}) - \nu \ln(1-\beta z^{-1})\right] = \\
&= -\nu \ln P(\beta) + \nu \ln P(0) - \nu^2 \ln(1-\beta^2) + \\
&\quad + \nu \oint dz \ln(1-\beta z) \frac{\partial}{\partial z} \ln Q(z^{-1}).
\end{aligned} \tag{4.5}$$

The last term in the right-hand side of (4.5) one integrates by parts

$$\nu \oint dz \ln(1-\beta z) \frac{\partial}{\partial z} \ln Q(z^{-1}) = \nu\beta \oint \frac{dz}{1-\beta z} \ln Q(z) = \nu \ln Q(\beta).$$

As a result we have

$$\ln\left(\frac{|\hat{V}^{(\nu)}(\beta)|}{|\hat{V}^{(\nu)}(0)|}\right) = -\nu \ln P(\beta) + \nu \ln Q(\beta) - \nu^2 \ln(1-\beta^2) = \ln\left[\frac{1-\beta^2}{(1-\alpha\beta)^2}\right]^{\nu^2}$$

For $(ma) = 0$ ($\alpha = 1$) matrix elements of the matrix

$$V^{(\nu)}_{x,x'}(0) = \oint \frac{dz}{z} z^{-(x-x')} V^{(\nu)}(z) \Big|_{\beta=0}$$

are calculated in the explicit form

$$V^{(\nu)}_{x,x'}(0) = \frac{\sin \pi\nu}{\pi} \frac{1}{\nu - x + x'} \tag{4.6}$$

Call to mind that x, x' take the values $0, \ldots, r-1$. To simplify notations let introduce

$$\hat{U}_r \equiv V^{(\nu)}_{x,x'}(0), \tag{4.7}$$

where the lower index for the matrix (4.6) indicates its dimension. Multiplying the matrix (4.6) from the left by the triangular matrix $\hat{L}_r$ and the right by the triangular matrix $\hat{R}_r$, we get the following relation

$$\hat{L}_r \cdot \hat{U}_r \cdot \hat{R}_r = \begin{pmatrix} \frac{\sin \pi\nu}{\pi} & \hat{0} \\ \hat{0} & -\hat{U}_{r-1} \end{pmatrix}, \tag{4.8}$$

where

$$\hat{L}_r = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ -\frac{1}{1} & \frac{\nu}{1} - 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{r-1} & 0 & \ldots & \frac{\nu}{r-1} - 1 \end{pmatrix}, \quad \hat{R}_r = \begin{pmatrix} \nu & -\frac{1}{1} & \ldots & -\frac{1}{r-1} \\ 0 & \frac{\nu}{1} + 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\nu}{r-1} + 1 \end{pmatrix}. \tag{4.9}$$

The determinant of a product of these matrises it is not hard to calculate

$$|\hat{L}_r \cdot \hat{R}_r| = \frac{(-1)^{r-1} \sin \pi\nu}{\pi} \cdot \frac{\Gamma(r+\nu)\Gamma(r-\nu)}{\Gamma^2(r)}. \tag{4.10}$$

This allows to get from (4.8) and (4.10) the simple recursion relation for the determinants

$$|\hat{U}_{r-1}| = |\hat{U}_r| \frac{\Gamma(r+\nu)\Gamma(r-\nu)}{\Gamma^2(r)},$$

which has the solution

$$|\hat{U}_r| = \prod_{k=1}^{r} \frac{\Gamma^2(k)}{\Gamma(k+\nu)\Gamma(k-\nu)}. \tag{4.11}$$

Using the following representation for Γ-function

$$\ln \Gamma(z) = \int_0^\infty \frac{dt}{t} \left(\frac{\mathrm{e}^{zt} - \mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} + (z-1)\mathrm{e}^{-t} \right),$$

the solution (4.11) can be represented in the form

$$\ln |\hat{U}_r| = -\int_0^\infty \frac{dt}{t} (1 - \mathrm{e}^{-tr}) \frac{\mathrm{sh}^2 \frac{\nu t}{2}}{\mathrm{sh}^2 \frac{t}{2}}. \tag{4.12}$$

For $r \to \infty$ the integral in the right-hand side of (4.12) is divergent for the zero. Rewriting (4.12) in the form

$$\ln|\hat{U}_r| = -\nu^2 \ln r - \int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2 \mathrm{e}^{-t}\right) + \int_0^\infty \frac{dt}{t}\mathrm{e}^{-tr}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2\right),$$

we can isolate this divergence. Now the integral depending on r is well defined and we get

$$\ln|\hat{U}_r| \underset{r\to\infty}{=} -\nu^2 \ln r - \int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2 \mathrm{e}^{-t}\right) + \mathrm{O}(1/r^2). \tag{4.13}$$

Taking into account (3.1), (4.3), (4.4), (4.7) and (4.13), $G(r)$ one finds the following asymptotic evaluations for the correlation function

$$G(r) \underset{r\to\infty,\,\alpha<1}{=} \left[\frac{(1-\alpha^2)(1-\beta^2)}{(1-\alpha\beta)^2}\right]^{2\nu^2} = (2\sqrt{2}ma)^{2\nu^2}, \tag{4.14}$$

$$G(r) \underset{r\to\infty,\,\alpha=1}{=} \left[\frac{(1-\beta^2)}{r(1-\alpha\beta)^2}\right]^{2\nu^2} \exp\left[-2\int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2 \mathrm{e}^{-t}\right)\right] = \frac{2^{\nu^2}\mathrm{e}^{-2\rho(\nu)}}{r^{2\nu^2}}, \tag{4.15}$$

where

$$\rho(\nu) = \int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2 \mathrm{e}^{-t}\right), \quad 2(ma) \underset{\alpha\to 1}{=} 1-\alpha^2.$$

Using these asymptotics one can assume that in the scaling limit ($r \to \infty$, $ma \to 0$, $mar = s = \mathrm{const}$) the correlation function $G(r)$ has the scaling form

$$F(s,\nu) = \lim r^{2\nu^2} G(r) = 2^{3\nu^2} s^{2\nu^2} f(s,\nu), \tag{4.16}$$

where the function $f(s,\nu)$ has the asymptotics

$$f(s,\nu) = \begin{cases} 1 \quad \text{for } s \to \infty, \\ 2^{-2\nu^2}\mathrm{e}^{-2\rho(\nu)} s^{-2\nu^2} \quad \text{for } s \to 0. \end{cases} \tag{4.17}$$

This scaling behaviour follows from the arquments:

1) the two-point scaling function $F(s,\nu)$ have to interpolate between the behaviour at large distance away from the critical temperature ($r \to \infty$, $ma \to 0$, so that $s \to \infty$) and the behaviour at large distance at the critical point ($r \to \infty$, $ma \to 0$, so that $s \to 0$), that is the asymptotics (4.17) have to result in asymptotics (4.14), (4.15).

2) for $\nu = \frac{1}{2}$ the scaling function (4.16) and the asymptotics (4.17) have to coincide with square of the scaling function and the asymptotics for the two-point correlation function of the disorder field in the Ising model [18, 19, 20].

It is convenient to define the ratio $h(Rm)$, which does not depend on a normalization and the lattice cutoff a,

$$h(Rm) = \frac{G(R)|_{R\to\infty}}{G(R)|_{R\to 0}} = \frac{F(s,\nu)|_{s\to\infty}}{F(s,\nu)|_{s\to 0}} = (2mR)^{2\nu^2}\exp\left[2\int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\frac{\nu t}{2}}{\mathrm{sh}^2\frac{t}{2}} - \nu^2 \mathrm{e}^{-t}\right)\right], \tag{4.18}$$

where $R = (ra)$ denotes a dimensional distance.

To compare this asymptotics with the result of the paper [13] in the free fermion point we use the following normalization for the correlation function for $R \to 0$ [13]

$$G(R) = R^{-2\nu^2}. \tag{4.19}$$

Using this normalization, from the ratio (4.18) we obtain

$$G(R) \underset{R\to\infty,\, m\neq 0}{=} \langle \mu_\nu \rangle^2 = (2m)^{2\nu^2} \exp\left[2 \int_0^\infty \frac{dt}{t} \left(\frac{\mathrm{sh}^2 \frac{\nu t}{2}}{\mathrm{sh}^2 \frac{t}{2}} - \nu^2 \mathrm{e}^{-t} \right) \right]. \tag{4.20}$$

5. Conclusion

In this work a determinant representation for the two-point correlation function of the twist field is obtained in the lattice fermion field model. The large distance behaviour of this correlation function is calculated and the vacuum expectation values of the twist fields is found. This value differs from the vacuum expectation value of the exponential field by the constant.

For explanation of this fact one can bring the following arguments. It is known that the quantum fields defined by the commutation relations (1.2) realize the operator solution of the isomodronomic deformation problem for the Dirac equation [11]. The correlation function of the twist fields (2.1) one can interpret as the functional integral in the Dirac fermion theory with given monodromy properties for fermion fields. Then this correlation function can be connected with other solution of the isomodronomic deformation problem for the Dirac equation. In order to check this asssumption it is necessary to calculate the determinant (2.9) and to obtain a differential equation for the correlation function of the twist fields.

Note that with our point of view in Appendix B of the paper [13] instead of the vacuum expectation values of the exponential fields the vacuum expectation values of the twist fields was calculated but since the normalization on behaviour of the correlation function of exponential field for $r \to 0$ was used the right result (1.8) for the vacuum expectation values of the exponential fields was obtained.

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6. Appendix A

In this Appendix we obtain the following representation for the determinant in (3.1) for $r \to \infty$

$$g(r) = |\cos^2 \pi\nu + \sin^2 \pi\nu \hat{C} \cdot \hat{C}^T| \underset{r\to\infty}{=} |1 - \sin^2 \pi\nu \hat{A}|^2 \tag{A.1}$$

where for convenience we introduced the matrix

$$C_{xy} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z} z^{-(x-y)} C(z), \quad C(z) = \left[\frac{(1-\alpha z)(1-\beta z^{-1})}{(1-\alpha z^{-1})(1-\beta z)}\right]^{\mu}, \tag{A.2}$$

It has dimension $(r+1)\times(r+1)$ $(x, y = 0, 1, \dots, r)$ and $0 < \mu < 1$. For $\mu = \frac{1}{2}$ this matrix coincides with the matrix $V_{x,y}$ in the determinant representation of the correlation function (2.9). In (A.1) the matrix $\hat{A}$ has the form

$$A_{x,y} = \oint\limits_{|z_1 z_2|<1} \frac{dz_1 dz_2 (z_1)^x (z_2)^y}{(1-z_1 z_2)} C^{-1}(z_1) C^{-1}(z_2).$$

Let us represent the product $\hat{C}\cdot\hat{C}^T$ in the following form

$$(\hat{C}\cdot\hat{C}^T)_{xy} = \oint_{|z|<|\xi|<1} \frac{d(z\xi)}{z\xi} z^{-x}\xi^y C(z) C(\xi^{-1}) \sum_{x'=0}^{r} \left(\frac{z}{\xi}\right)^{x'} =$$

$$\oint_{|\xi|<1<|z|} \frac{d(z\xi) z^{-x}\xi^y}{z\xi(1-z/\xi)} \frac{C(z)}{C(\xi)} + \oint \frac{dz}{z} z^{-x+y} - \oint_{|z|<|\xi|<1} \frac{d(z\xi) z^{r+1-x}\xi^{y-r-1}}{z\xi(1-z/\xi)} \frac{C(z)}{C(\xi)} =$$

$$\delta(x-y) - A_{xy} - B_{xy}, \tag{A.3}$$

where

$$A_{xy} = -\oint_{|\xi|<1<|z|} \frac{d(z\xi) z^{-x}\xi^y}{z\xi(1-z/\xi)} \frac{C(z)}{C(\xi)} \underset{z\to 1/z}{=} \oint \frac{d(z\xi) z^x \xi^y}{1-z\xi} \frac{1}{C(z)C(\xi)}$$

$$B_{xy} = \oint_{|z|<|\xi|<1} \frac{d(z\xi) z^{r+1-x}\xi^{y-r-1}}{z\xi(1-z/\xi)} \frac{C(z)}{C(\xi)} \underset{\xi\to 1/\xi}{=} \oint \frac{d(z\xi) z^{r+1-x}\xi^{y-r-1}}{1-z\xi} C(z) C(\xi)$$

In (A.3) we summed a geometric progression over x' and used $C(z^{-1}) = C^{-1}(z)$. Substituting (A.3) in (A.1), we get

$$g(r) = |1 - \sin^2 \pi\nu(\hat{A} + \hat{B})| = |(1 - s^2 \hat{A}) \cdot (1 - s^2 \hat{B}) - s^4 \hat{A}\cdot\hat{B}|, \tag{A.4}$$

where $s = \sin \pi\nu$.

Note that matrix elements of the matrices $\hat{A}$ and $\hat{B}$ have the following asymptotic behaviour

$$A_{xy} \underset{x+y\gg 1}{\sim} \alpha^{x+y}, \quad B_{xy} \underset{2r-x-y\gg 1}{\sim} \alpha^{2r-x-y}.$$

and therefore the product $\hat{A}\cdot\hat{B}$ is a exponential small matrix. Really,

$$(\hat{A}\cdot\hat{B})_{xy} = \oint \frac{d(z_1...z_4)}{(1-z_1 z_2)(1-z_3 z_4)} \frac{C(z_3)C(z_4)}{C(z_1)C(z_2)} (z_1)^x (z_4)^{r-y} \sum_{x'=0}^{r} (z_2)^{x'} (z_3)^{r-x'} =$$

$$\oint \frac{d(z_1...z_4)}{(1-z_1 z_2)(1-z_3 z_4)} \frac{C(z_3)C(z_4)}{C(z_1)C(z_2)} (z_1)^x (z_4)^{r-y} \frac{(z_3)^{r+1} - (z_2)^{r+1}}{z_3(1-z_2/z_3)} \underset{r\to\infty}{\sim}$$

$$\frac{\alpha^r \ln r}{r} \left(\oint \frac{dz z^x}{1-\alpha z} C(z^{-1})\right) \left(\oint \frac{dz z^{r-y}}{1-\alpha z} C(z)\right) \tag{A.5}$$

Taking into account (A.5) in (A.4), we obtain at $r \to \infty$

$$g(r) \underset{r\to\infty}{=} |(1 - s^2\hat{A})| \cdot |(1 - s^2\hat{B})|. \tag{A.6}$$

Let us show that $|(1 - s^2\hat{A})| = |(1 - s^2\hat{B})|$ for $r \to \infty$. For this we write the matrix $\hat{B}$ in the form

$$\hat{B} = \hat{J} \cdot \hat{D} \cdot \hat{J},$$

where

$$J_{xy} = \delta(x + y - r) = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}, \quad (\hat{J} \cdot \hat{J})_{xy} = \delta(x - y),$$

$$D_{xy} = \oint \frac{d(z\xi) z^x \xi^y}{1 - z\xi} C(z)C(\xi). \tag{A.7}$$

Using this representation, it is not hard to show that

$$|(1 - s^2\hat{A})| = |(1 - s^2\hat{D})| \tag{A.8}$$

The determinants in (A.8) it is convenient to decompose in the series

$$\ln|1 - s^2\hat{A}| = \mathrm{Sp}\ln\left(1 - s^2\hat{A}\right) = -\sum_{k=1}^{\infty} \frac{s^{2k}}{k} a_{2k},$$

$$\ln|1 - s^2\hat{D}| = \mathrm{Sp}\ln\left(1 - s^2\hat{D}\right) = -\sum_{k=1}^{\infty} \frac{s^{2k}}{k} d_{2k},$$

where

$$a_{2k} = \mathrm{Sp}\left(\hat{A}\right)^k, \quad d_{2k} = \mathrm{Sp}\left(\hat{D}\right)^k.$$

For $r \to \infty$ the matrix $\hat{A}$ and $\hat{D}$ can be represented in the factorized form

$$\hat{A} \underset{r\to\infty}{=} \hat{F}^{(a)} \cdot \hat{F}^{(a)}, \quad \hat{D} \underset{r\to\infty}{=} \hat{F}^{(d)} \cdot \hat{F}^{(d)}, \quad F^{(a)}_{xy} = \oint dz z^{x+y} C(z), \quad F^{(d)}_{xy} = \oint dz z^{x+y} C^{-1}(z),$$

so that

$$a_{2k} = \left(\mathrm{Sp}\hat{F}^{(a)}\right)^{2k} = \oint \prod_{l=1}^{2k} \left(\frac{dz_l C^{-1}(z_l)}{1 - z_l z_{l+1}}\right), \quad z_{2k+1} \equiv z_1,$$

$$d_{2k} = \left(\mathrm{Sp}\hat{F}^{(d)}\right)^{2k} = \oint \prod_{l=1}^{2k} \left(\frac{dz_l C(z_l)}{1 - z_l z_{l+1}}\right), \quad z_{2k+1} \equiv z_1. \tag{A.9}$$

Consider, for example, the case $k = 1$

$$d_2 = \oint_{|z_1 z_2|<1} \frac{dz_1 dz_2}{(1 - z_1 z_2)^2} C(z_1)C(z_2) \underset{z_2\to 1/z_2}{=} \oint_{|z_1|<|z_2|} \frac{dz_1 dz_2}{(z_1 - z_2)^2} \frac{C(z_1)}{C(z_2)} =$$

$$\oint_{|z_2|<|z_1|} \frac{dz_1 dz_2}{(z_1 - z_2)^2} \frac{C(z_1)}{C(z_2)} + \oint dz C(z) \frac{\partial}{\partial z} C^{-1}(z). \tag{A.10}$$

Such as in the right-hand side of (A.10) the second term is a integral over a closed contour from a derivative it is equal to zero

$$\oint dzC(z)\frac{\partial}{\partial z}C^{-1}(z) = -\oint dz\frac{\partial}{\partial z}\ln C(z) = 0.$$

In the first term in the right-hand side of (A.10) we did the replacement $z_1 \to (z_1)^{-1}$

$$d_2 = \oint_{|z_1z_2|<1}\frac{dz_1dz_2}{(1-z_1z_2)^2}\frac{1}{C(z_1)C(z_2)} = a_2. \qquad (A.11)$$

For $k > 1$ it is not hard to get the recurent relation

$$d_{2k} = a_{2k} + d_{2k-2} - a_{2k-2}. \qquad (A.12)$$

For this in the integral representation (A.9) it is necessary to do the replacement $z_l \to (z_l)^{-1}$. After that we obtain new integration contours with $|z_lz_{l+1}| > 1$. Contracting sequentially these contours over the variables z_l so that $|z_lz_{l+1}| < 1$ and calculating residues in the poles $z_l = (z_{l+1})^{-1}$, we get (A.12).

Taking into account the "initial condition" (A.11), it is easy to check that a solution of the recurent relation (A.12) is $a_{2k} = d_{2k}$. From here we obtain (A.8). Then (A.8) and (A.6) lead to (A.1).

Note that putting $\mu = \frac{1}{2}$ in (A.2), we get (3.3) from (A.1), and putting $\nu = h$ in (A.1) and $\mu = \nu$ in (A.2), we get (3.17) from (A.1).

7. Appendix B

In this Appendix we obtain the asymptotic evaluation of $|\hat{V}^{(\nu)}|$ for $r \to \infty$

$$|\hat{V}^{(\nu)}| \underset{r\to\infty}{=} \left[\frac{(1-\alpha^2)(1-\beta^2)}{(1-\alpha\beta)}\right]^{\nu^2}. \qquad (B.1)$$

It is convenient to introduce the following notations

$$V^{(\nu)}_{xy} \equiv B_{xy} = \oint\frac{dz}{z}z^{-x+y}B(z), \quad B(z) = P(z)Q(z^{-1}), \qquad (B.2)$$

where $x, y = 0, 1, ..., r$, the functions $P(z)$ and $Q(z)$ is anylitical in the circle $|z| \leq 1$

$$P(z) = \left(\frac{1-\alpha z}{1-\beta z}\right)^{\nu}, \quad Q(z) = \left(\frac{1-\beta z}{1-\alpha z}\right)^{\nu} = \frac{1}{P(z)}, \quad \alpha > \beta \leq 0, \qquad (B.3)$$

α and β are determined in (2.11).

Let us define

$$B_{xy}(\mu) = \oint\frac{dz}{z}z^{-x+y}B^{\mu}(z), \quad B^{\mu}(z) = P^{\mu}(z)Q^{\mu}(z^{-1}), \qquad (B.4)$$

and

$$f(\mu) = \ln|\hat{B}(\mu)|, \quad f(0) = 0, \; f(1) = \ln|\hat{B}|, \; B_{xy}(1) = B_{xy}, \; B_{xy}(0) = \delta(x-y),$$

For calculation of $|\hat{B}|$ we use the formula

$$\ln|\hat{B}| = f(1) = \int_0^1 f'(\mu), \quad f'(\mu) = \mathrm{Sp}\hat{B}'(\mu)\cdot\hat{B}^{-1}(\mu). \tag{B.5}$$

Let us find the inverse matrix $\hat{B}^{-1}(\mu)$. Note that the matrix $\hat{B}(\mu)$ for $0 \le \mu \le 1$ has the same analytic behaviour as the matrix $\hat{B}$, therefore, for simplicity we only derive the inverse matrix $\hat{B}^{-1}$.

Show that matrix $\hat{D}$

$$\hat{D} = \hat{L} - \hat{E} - \hat{J}\cdot\hat{E}^T\cdot\hat{J} \tag{B.6}$$

is inverse to $\hat{B}$ up to an exponentially small term

$$\hat{B}\cdot\hat{D} = I + \mathrm{O}(\alpha^r) \quad \text{at } r\to\infty,$$

where $J^{xy} = \delta(x+y-r)$,

$$E_{xy} = \oint_{|z_1z_2|<1}\frac{dz_1dz_2}{1-z_1z_2}\frac{(z_1)^x(z_2)^y}{P(z_1^{-1})Q(z_2^{-1})} = \oint_{|z_2|<|z_1|}\frac{dz_1dz_2}{z_1(z_1-z_2)}\frac{(z_1)^{-x}(z_2)^y}{P(z_1)Q(z_2^{-1})}, \tag{B.7}$$

$$L_{xy} = \oint\frac{dz}{z}\frac{z^{-x+y}}{B(z)} = B^T_{xy}, \quad \text{if } B(z^{-1}) = B^{-1}(z),$$

since

$$B^T_{xy} = B_{yx} = \oint\frac{dz}{z}z^{-y+x}B(z) \underset{z\to1/z}{=} \oint\frac{dz}{z}z^{-x+y}B(z^{-1})$$

and in our case $B(z^{-1}) = B^{-1}(z)$.

At the first we calculate $\hat{B}\cdot\hat{L}$

$$\left(\hat{B}\cdot\hat{L}\right)_{xy} = \oint_{|z_1|<|z_2|}\frac{dz_1dz_2B(z_1)}{z_1z_2B(z_2)}\frac{1-(z_1/z_2)^{r+1}}{1-z_1/z_2}(z_1)^{-x}(z_2)^y = \delta(x-y) -$$

$$\oint_{|z_1z_2|<1}\frac{dz_1dz_2B(z_1^{-1})}{z_1z_2B(z_2)}(z_1)^x(z_2)^y - \oint_{|z_1z_2|<1}\frac{dz_1dz_2B(z_1)}{z_1z_2B(z_2^{-1})}(z_1)^{r-x}(z_2)^{r-y}. \tag{B.8}$$

Now one calculates the product $\hat{B}\cdot\hat{E}$

$$\left(\hat{B}\cdot\hat{E}\right)_{xy} = \oint_{|z_1|<|z_3|<|z_2|}\frac{d(z_1z_3)(z_1)^{-x}(z_3)^y}{z_1z_2(z_2-z_3)}\frac{B(z_1)}{P(z_2)Q(z_3^{-1})}\frac{1-(z_1/z_2)^{r+1}}{1-z_1/z_2} =$$

$$\oint_{|z_1|<|z_3|}\frac{d(z_1z_3)(z_1)^{-x}(z_3)^y}{z_1(z_3-z_1)}\frac{B(z_1)}{B(z_3)} + \oint_{|z_1|<|z_3|}\frac{d(z_1z_3)(z_1)^{-x}(z_3)^y}{z_1(z_1-z_3)}\frac{Q(z_1^{-1})}{Q(z_3^{-1})} -$$

$$\oint_{\alpha<|z_i|<1}\frac{d(z_1z_2z_3)(z_1)^{r-x}(z_3)^y(z_2)^{r+1}}{(1-z_1z_2)(1-z_2z_3)}\frac{B(z_1)}{P(z_2^{-1})Q(z_3^{-1})}. \tag{B.9}$$

In (B.9) the integration over z_2 yields a contribution which is propotional to α^{r+1} and therefore, the last term in the right-hand side of (B.9) is exponentially small for any x, y.

The second term in the right-hand side of (B.9) gives the following contribution at the contraction of the contour over z_3

$$\oint_{|z_3|<|z_1|} \frac{d(z_1 z_3)(z_1)^{-x}(z_3)^y}{z_1(z_1-z_3)} \frac{Q(z_1^{-1})}{Q(z_3^{-1})} - \delta(x-y) = -\delta(x-y), \tag{B.10}$$

where we took into account that the first term in the left-hand side of (B.10) reduces to zero at the extention of the contour over z_3 to one of infinite radius.

Similarly for the first term in the right-hand side of (B.9) one gets

$$\oint_{|z_3|<|z_1|} \frac{d(z_1 z_3)(z_1)^{-x}(z_3)^y}{z_1(z_3-z_1)} \frac{B(z_1)}{B(z_3)} + \delta(x-y). \tag{B.11}$$

Taking into account (B.10) and (B.11), we obtain

$$\begin{aligned}\left(\hat{B}\cdot\hat{E}\right)_{xy} &= -\oint_{|z_2|<|z_1|} \frac{d(z_1 z_2)(z_1)^{-x}(z_2)^y}{z_1(z_2-z_1)} \frac{B(z_1)}{B(z_2)} - \\ \oint_{\alpha<|z_i|<1} &\frac{d(z_1 z_2 z_3)(z_1)^{r-x}(z_3)^y(z_2)^{r+1}}{(1-z_1 z_2)(1-z_2 z_3)} \frac{B(z_1)}{P(z_2^{-1})Q(z_3^{-1})} = \\ \underset{z_1\to 1/z_1}{=} &- \oint_{|z_1 z_2|<1} \frac{d(z_1 z_2)(z_1)^x(z_2)^y}{(1-z_1 z_2)} \frac{B(z_1^{-1})}{B(z_2)} - \mathrm{O}(\alpha^r).\end{aligned} \tag{B.12}$$

From here it follows that $\hat{B}\cdot\hat{E}$ coincides up to $\mathrm{O}(\alpha^r)$ with the second term in the right-hand side of (B.8) of the product $\hat{B}\cdot\hat{L}$.

Now let us consider the product $\hat{B}\cdot\hat{J}\hat{E}^T\hat{J}$

$$\hat{B}\cdot\hat{J}\hat{E}^T\hat{J} = \hat{J}\cdot\left(\hat{J}\cdot\hat{B}\cdot\hat{J}\cdot\hat{E}^T\right)\cdot\hat{J} = 0\hat{J}\cdot\left(\hat{B}^T\cdot\hat{E}^T\right)\cdot\hat{J}, \tag{B.13}$$

where we used $\hat{J}\cdot\hat{B}\cdot\hat{J} = \hat{B}^T$.

A transition from $\hat{B}$, $\hat{E}$ to $\hat{B}^T$, $\hat{E}^T$ means the following transformations in contour integrals (B.2) and (B.7): $B(z)\to B(z^{-1})$, $P(z)\leftrightarrow Q(z)$ and therefore, no calculating, we have instead of (B.12)

$$\begin{aligned}\left(\hat{B}^T\cdot\hat{E}^T\right)_{xy} &= -\oint_{|z_1 z_2|<1} \frac{d(z_1 z_2)(z_1)^x(z_2)^y}{(1-z_1 z_2)} \frac{B(z_1)}{B(z_2^{-1})} - \\ \oint_{\alpha<|z_i|<1} &\frac{d(z_1 z_2 z_3)(z_1)^{r-x}(z_3)^y(z_2)^{r+1}}{(1-z_1 z_2)(1-z_2 z_3)} \frac{B(z_1^{-1})}{Q(z_2^{-1})P(z_3^{-1})}\end{aligned} \tag{B.14}$$

and from (B.13) and (B.14) one gets

$$\hat{B}\cdot\hat{J}\hat{E}^T\hat{J} = -\oint_{|z_1 z_2|<1} \frac{d(z_1 z_2)(z_1)^{r-x}(z_2)^{r-y}}{(1-z_1 z_2)} \frac{B(z_1)}{B(z_2^{-1})} - \mathrm{O}(\alpha^r). \tag{B.15}$$

Collecting together (B.8), (B.12) and (B.15), one proves that $\hat{D} = \hat{B}^{-1} + \mathrm{O}(\alpha^r)$. Thus for $f'(\mu)$ in (B.5) we obtain

$$f'(\mu) = \mathrm{Sp}\hat{B}'(\mu)\cdot\left[\hat{L}(\mu) - \hat{E}(\mu) - \hat{J}\cdot\hat{E}^T(\mu)\cdot\hat{J}\right]. \tag{B.16}$$

Note that

$$\mathrm{Sp}\hat{B}'\cdot\hat{J}\cdot\hat{E}^T\cdot\hat{J}=\mathrm{Sp}\hat{J}\cdot\hat{B}'\cdot\hat{J}\cdot\hat{E}^T=\mathrm{Sp}\left(\hat{B}'\right)^T\cdot\hat{E}^T=\mathrm{Sp}\left(\hat{B}'\right)^T\cdot\hat{E}^T=\mathrm{Sp}\hat{B}'(\mu)T\cdot\hat{E}(\mu).$$

From here one gets

$$f'(\mu)=\mathrm{Sp}\hat{B}'(\mu)\left(\hat{L}(\mu)-2\hat{E}(\mu)\right).$$

Let us denote by

$$U_1=\mathrm{Sp}\left(\hat{B}'(\mu)\hat{L}(\mu)\right),\quad U_2=\mathrm{Sp}\left(\hat{B}'(\mu)\hat{E}(\mu)\right),\quad f'(\mu)=U_1-2U_2. \tag{B.17}$$

At the first we calculate U_1

$$U_1=\sum_{x,y}\oint\frac{dz_1dz_2}{z_1z_2}z_1^{-y+x}B^{\mu}(z_1)\ln B(z_1)z_2^{-x+y}B^{-\mu}(z_2)=$$

$$\oint_{|z_1|<|z_2|}\frac{dz_1dz_2}{z_1z_2}\frac{B^{\mu}(z_1)\ln B(z_1)}{B^{\mu}(z_2)}\frac{[1-(z_1/z_2)^{r+1}][1-(z_2/z_1)^{r+1}]}{(1-z_1/z_2)(1-z_2/z_1)}=$$

$$(r+1)\oint\frac{dz}{z}\ln B(z)-\left(\oint_{|z_1|<|z_2|}+\oint_{|z_2|<|z_1|}\right)\frac{dz_1dz_2}{(z_1-z_2)^2}\frac{B^{\mu}(z_1)\ln B(z_1)}{B^{\mu}(z_2)}+$$

$$\oint_{|z_1|<|z_2|}\frac{dz_1dz_2}{(z_1-z_2)^2}\frac{B^{\mu}(z_1)\ln B(z_1)}{B^{\mu}(z_2)}\left(\frac{z_1}{z_2}\right)^{r+1}+\oint_{|z_2|<|z_1|}\frac{dz_1dz_2}{(z_1-z_2)^2}\frac{B^{\mu}(z_1)\ln B(z_1)}{B^{\mu}(z_2)}\left(\frac{z_2}{z_1}\right)^{r+1}.$$

In this expression the last two terms are exponentially small ($\sim \mathrm{O}(\alpha^r)$)) and therefore,

$$U_1=(r+1)\oint\frac{dz}{z}\ln B(z)-\left(\oint_{|z_1|<|z_2|}+\oint_{|z_2|<|z_1|}\right)\frac{dz_1dz_2}{(z_1-z_2)^2}\frac{B^{\mu}(z_1)\ln B(z_1)}{B^{\mu}(z_2)}+\mathrm{O}(\alpha^r) \tag{B.18}$$

Now let us calculate U_2

$$U_2=\sum_{x,y}\oint\frac{d(z_1z_2z_3}{z_1(1-z_2z_3)}z_1^{-y+x}z_2^yz_3^x\frac{B^{\mu}(z_1)\ln B(z_1)}{P^{\mu}(z_2^{-1})Q^{\mu}(z_3^{-1})}=$$

$$\oint_{|z_3|<|z_2|<|z_1|<1}\frac{d(z_1z_2z_3}{z_1(1-z_2z_3)}\frac{B^{\mu}(z_1)\ln B(z_1)}{P^{\mu}(z_2^{-1})Q^{\mu}(z_3^{-1})}\frac{[1-(z_1z_2)^{r+1}][1-(z_3/z_1)^{r+1}]}{(1-z_1z_2)(1-z_3/z_1)}=$$

$$\oint_{|z_3|<|z_1|<|z_2|}\frac{d(z_1z_2z_3}{(z_1-z_3)(z_2-z_1)(z_2-z_3)}\frac{B^{\mu}(z_1)\ln B(z_1)}{P^{\mu}(z_2)Q^{\mu}(z_3^{-1}])}+\mathrm{O}(\alpha^r). \tag{B.19}$$

In the right-hand side of (B.19) contracting the contour over z_2, we obtain

$$\oint_{|z_3|<|z_1|}\frac{d(z_1z_3}{(z_1-z_3)^2}\frac{Q^{\mu}(z_1^{-1})\ln B(z_1)}{Q^{\mu}(z_3^{-1}])}-\oint_{|z_3|<|z_1|}\frac{d(z_1z_3}{(z_1-z_3)^2}\frac{B^{\mu}(z_1)\ln B(z_1)}{B^{\mu}(z_3)}+\mathrm{O}(\alpha^r). \tag{B.20}$$

In the first term of the right-hand side of (B.20) let us move the contour over z_3 to ∞. Then

$$U_2=\oint dzQ^{\mu}(z^{-1})\left[\ln P(z)+\ln Q(z^{-1})\right]\frac{\partial}{\partial z}\left(\frac{1}{Q^{\mu}(z^{-1})}\right)=$$

$$\mu \oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}) + \mathrm{O}(\alpha^r). \tag{B.21}$$

Substituting (B.18) and (B.21) in (B.17), one gets

$$f'(\mu) = (r+1) \oint \frac{dz}{z} \ln B(z) - \left(\oint_{|z_1|<|z_2|} - \oint_{|z_2|<|z_1|} \right) \frac{dz_1 dz_2}{(z_1 - z_2)^2} \frac{B^\mu(z_1) \ln B(z_1)}{B^\mu(z_2)} -$$

$$-2\mu \oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}) + \mathrm{O}(\alpha^r). \tag{B.22}$$

The difference of the integrals over z_2 in the second term of the right-hand side of (B.22) is a integral over z_2 around the point z_1, that is

$$\oint_{(z_1)} \frac{dz_2}{(z_1 - z_2)^2} \frac{1}{B^\mu(z_2)} = \frac{\partial}{\partial z} \left(\frac{1}{B^\mu(z)} \right)_{z=z_1}.$$

Then the second term in the right-hand side of (B.22) has the form

$$\mu \oint dz \ln B(z) \frac{\partial}{\partial z} \ln B(z) = \frac{\mu}{2} \int dz \frac{\partial}{\partial z} \left(\ln^2 B(z) \right) = 0,$$

since the integral over a closed contour from a total derivative is equal to the zero.

Thus

$$f'(\mu) = (r+1) \oint \frac{dz}{z} \ln B(z) - -2\mu \oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}). \tag{B.23}$$

Integrating this expression over μ in the limits $[0,1]$, we obtain

$$f(1) = \ln |\hat{B}| = (r+1) \oint \frac{dz}{z} \ln B(z) - \oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}). \tag{B.24}$$

Substituting here the explicit expressions for $B(z)$, $P(z)$ and $Q(z)$ from (B.2) and (B.3) we have

$$\oint \frac{dz}{z} \ln B(z) = \nu \left(\oint \frac{dz}{z} \ln P(z) - \oint \frac{dz}{z} \ln P(z^{-1}) \right) = \nu \left(P(0) - P(0) \right) = 0,$$

$$-\oint dz \ln P(z) \frac{\partial}{\partial z} \ln Q(z^{-1}) = \nu^2 \oint dz \ln P(z) \left(\frac{1}{z-\alpha} - \frac{1}{z-\beta} \right) = \mu^2 \left(\ln P(\alpha) - \ln P(\beta) \right).$$

Thus

$$|\widehat{V}^{(\nu)}| = |\hat{B}| \underset{r\to\infty}{=} \left(\frac{P(\alpha)}{P(\beta)} \right)^{\nu^2} = \left[\frac{(1-\alpha^2)(1-\beta^2)}{(1-\alpha\beta)} \right]^{\nu^2}. \tag{B.25}$$

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